Q.2 a. Show that for any two sets A and B, A - B = A - (A $\cap$ B).

Answer:

1et $x \in (A - B)$

$\Rightarrow x \in A$ and $x \notin B$

$\Rightarrow x \in A$ and $x \notin (A \cap B)$

$\Rightarrow x \in A - (A \cap B)$

Thus, $(A - B) \subseteq A - (A \cap B)$ …… (1)

Again, let $x \in A - (A \cap B)$

$\Rightarrow x \in A$ and $x \notin (A \cap B)$

$\Rightarrow x \in A$ and ($x \in A$ and $x \notin B$)

$\Rightarrow x \in A$ and $x \notin B$

$\Rightarrow x \in (A - B)$

Thus, $A - (A \cap B) \subseteq (A - B)$ ……. (2)

From (1) and (2), $A - B = A - (A \cap B)$.

b. The probability that A hits a target is 1/3 and the probability that B hits a target is 1/5. They both fire at the target. Find the probability that:

(i) A does not hit the target.
(ii) Both hit the target.
(iii) One of them hits the target.
(iv) Neither hits the target.

Answer:

We are given p(A) = 1/3 AND P(B) = 1/5 and we assume that they are independent.

(i) Probability that A does not hit the target = $p(A^c) = 1 - p(A) = 2/3$.

(ii) Probability that both hit the target = $p(A \cap B) = p(A).p(B) = 1/15$.

(iii) Probability that one of them hits the target = $p(A \cup B)$
$p(A \cup B) = p(A) + p(B) - p(A \cap B) = 7/15$.

(iv) Probability that neither hits the target = $(p(A \cup B))^c = 1 - p(A \cup B) = 8/15$.

Q.3 a. Show that B→ E is a valid conclusion drawn from the following premises:

$A \vee (B \rightarrow D)$, $\sim C \rightarrow (D \rightarrow E)$, $A \rightarrow C$ and $\sim C$.

Answer:

Given that $A \vee (B \rightarrow D)$

$\Rightarrow \sim A \rightarrow (B \rightarrow D)$ (Because $p \rightarrow q \equiv \sim p \vee q$) …..(1)

Also given $\sim C$ and $A \rightarrow C$

$\Rightarrow \sim C$ and $\sim C \rightarrow \sim A$ ($p \rightarrow q \equiv \sim q \rightarrow \sim p$, Law of contrapositive)

$\Rightarrow \sim A$ (Law of detachment)

Now $\sim A$ and $\sim A \rightarrow (B \rightarrow D)$ (by 1)

$\Rightarrow B \rightarrow D$ (Law of detachment) ……(2)

Again by $\sim C$ and $\sim C \rightarrow (D \rightarrow E)$

$\Rightarrow D \rightarrow E$ (Law of detachment) …….(3)

By 2 and 3 we have $B \rightarrow D$ and $D \rightarrow E$

We have $B \rightarrow E$ (Law of transitivity)

which is the desired conclusion.

b. Express the statement $(\sim(p \vee q)) \vee ((\sim p) \wedge q)$ in simplest possible form.

Answer:

Given $(\sim(p \vee q)) \vee ((\sim p) \wedge q) = (\sim(p \vee q) \vee \sim p) \wedge (\sim(p \vee q) \vee q)$ (Distributive Law)

$= ((\sim p \wedge \sim q) \vee \sim p) \wedge ((\sim p \wedge \sim q) \vee q)$ (De'Morgan's Law)

$= ((\sim p \vee \sim p) \wedge (\sim q \vee \sim p)) \wedge ((\sim p \vee q) \wedge (\sim q \vee q))$ (Distributive Law)

$= (\sim p \wedge (\sim q \vee \sim p)) \wedge ((\sim p \vee q) \wedge T)$ ($\sim p \vee p = T$ and $p \wedge T = p$)

$= (\sim p \wedge [(\sim q \vee \sim p) \wedge (\sim p \vee q)]) = (\sim p \wedge [(\sim q \vee \sim p) \wedge (q \vee \sim p)])$

$= (\sim p \wedge [(\sim q \wedge q) \vee \sim p])$ (By right distributive law)

$= (\sim p \wedge [F \vee \sim p])$ ($\sim p \wedge p = F$ and $p \vee F = p$)

$= (\sim p \wedge \sim p)$

$= \sim p$.

Q.4 a. Show that $\neg \forall x(P(x) \rightarrow Q(x))$ and $\exists x(P(x) \wedge \neg Q(x))$ are logically equivalent.

Answer:

We have $\neg \forall x(P(x) \rightarrow Q(x)) \equiv \exists x(\sim(P(x) \rightarrow Q(x)))$ (Negation)

$\equiv \exists x(\sim(\sim P(x) \vee Q(x)))$ (by, $p \rightarrow q \equiv \sim p \vee q$)

$\equiv \exists x(P(x) \wedge \sim Q(x))$ (by De'Morgan's Law)

b. There are two restaurants next to each other. One has a sign that says, "Good food is not cheap" and the other has a sign that says, "Cheap food is not good". Are the signs saying the same thing?

Answer:

Let p: Food is good and q: Food is cheap.

Then "Good food is not cheap': $p \rightarrow \sim q$ …… (1)

Now "Cheap food is not good": $q \rightarrow \sim p$

Taking the contra positive, we have $p \rightarrow \sim q$ which is same as in (1)

Hence both the signs says same thing.

c. Verify that $[p\rightarrow (q\rightarrow r)] \rightarrow [(p\rightarrow q) \rightarrow (p\rightarrow r)]$ is a tautology.

Answer:

p	**q**	**r**	**A=(p→q)**	**B=(p→r)**	**C=(q→r)**	**D=p→C**	**E=A→B**	**D→E**
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	T	F	F	T	T	T	T
T	F	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	F	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	T	T	T	T

Hence it is a tautology.

Q.5 a. Let A = {1, 2, 3, 4, 5, 6} and let R be the relation defined by "x divides y" written as x / y.

(i) Write R as a set of ordered pairs.
(ii) Draw its directed graph.
(iii) Find R^{-1}.

Answer:

(i) According to the given relation R can be defined as

R = {(x, y) | x divides y, for all x and y in A}

Thus, R = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (5, 5), (6, 6).

(ii) Directed graph of relation R:

(iii) R^{-1} = {(1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1), (2, 2), (4, 2), (6, 2), (3, 3), (6, 3), (4, 4), (5, 5), (6, 6)}.

b. Prove that $n! \geq 2^n$ for $n \geq 4$, by using the principle of mathematical induction.

Answer:

Let P(n) be the predicate $n! \geq 2^n$.

Basic Step:- P(4) : $4! = 24 \geq 16 = 2^4$, hence P(4) is true.

Induction Step:- Let us assume that P(k) is true, i.e; $k! \geq 2^k$

Consider P(k+1) then $(k+1)! = (k+1).k!$ …… (1)

Now since $k \geq 4$

$\Rightarrow (k+1) \geq 4+1 = 5 > 2$.

Also, $k! \geq 2^k$ (by induction hypothesis)

Thus $(k + 1)! = (k + 1).k! \geq 2.2^k = 2^{(k + 1)}$

Hence proved.

Q.6 a. Let I be the set of integers and R be a binary relation defined on set I as R = {(x, y)| x ≡ y (mod 3), x є I, y є I}, show that R is an equivalence relation.

Answer:

$x \equiv y \pmod 3$ can be written as $x - y = 3k$, where k is any integer.

Then $R = \{(x, y) | x - y = 3k, x \in I, y \in I \text{ and } k \in I\}$

1. **<u>Reflexive:</u>** For any $x \in I$, $x - x = 0$
   Hence, x – x is divisible by 3. Therefore R is a reflexive relation.

2. **<u>Symmetric:</u>** for any $x, y \in I$ $(x, y) \in R \Rightarrow x - y = 3k$ (k is an integer)
   - ⇨ $y - x = 3(-k)$ (-k is an integer)
   - ⇨ $(y, x) \in R$

   Hence, R is a symmetric relation.

3. **<u>Transitive:</u>** For all $x, y, z \in I$, $(x, y) \in R$ and $(y, z) \in R$
   - ⇨ $x - y = 3k_1$ and $y - z = 3k_2$ (k_1 and k_2 are integers)
   - ⇨ $(x - y) + (y - z) = 3(k_1 + k_2) = 3m$ ($m = k_1 + k_2$ is an integer)
   - ⇨ $(x - z) = 3m$
   - ⇨ $(x, z) \in R$

   Hence R is transitive.

Since R is reflexive, symmetric and transitive relation, hence R is an equivalence relation.

b. Prove that if L is a bounded distributive lattice and if a complement exists in L, it is unique.

Answer:

Let L be a bounded and distributive lattice. Let $a \in L$. Let a1 and a2 be two complements of a. Then $a \vee a1 = I$ and $a \wedge a1 = 0$ and also $a \vee a2 = I$ and $a \wedge a2 = 0$

Then $a1 = a1 \vee 0 = a1 \vee (a \wedge a2) = (a1 \vee a) \wedge (a1 \vee a2)$ (by distributive law)

$= I \wedge (a1 \vee a2) = (a1 \vee a2)$ ………(1)

Also $a2 = a2 \vee 0 = a2 \vee (a \wedge a1) = (a2 \vee a) \wedge (a2 \vee a1)$

$= I \wedge (a2 \vee a1) = (a1 \vee a2)$ ………(2)

By (1) and (2) we have a1 = a2. Hence complements are unique.

Q.7 a. Consider the functions f : R→R and g : R→R, defined by $f(x) = 2x + 3$ and $g(x) = x^2 + 1$. Find the composite functions (gof)(x) and (fog)(x).

Answer:

Composite function: Consider a function f: A→B such that f(a) = b and g: B→C such that g(b) = c, then a function gof: A→C is called composite function and defined as

$(gof)(a) = g(f(a)) = [\, c \mid \exists$ an element $b \in B$ such that $f(a) = b$ and $g(b) = c]$.

We know that $(gof)(x) = g(f(x)) = g(2x+3) = (2x+3)^2 = 4x^2 + 12x + 9$.

and $(fog)(x) = f(g(x)) = f(x^2) = 2x^2 + 3$.

b. Let X = {a, b, c}. Define f : X→X such that f = {(a, b), (b, a), (c, c)}. Find:

(i) f^{-1}
(ii) f^2
(iii) f^3
(iv) f^4

Answer:

Since given function is one-one and onto, we can find it's inverse as

f^{-1} = {(b, a),(a, b), (c, c)}

f^2 = fof = {(a, b), (b, a), (c, c)}o{(a, b), (b, a), (c, c)}= {(a, a), (b, b), (c, c)}
f^3 = fof^2 = {(a, b), (b, a), (c, c)}o{(a, a), (b, b), (c, c)}= {(a, b), (b, a), (c, c)}
f^4 = fof^3 = {(a, b), (b, a), (c, c)}o{(a, b), (b, a), (c, c)}= {(a, a), (b, b), (c, c)}

Q.8 a. Show that the set of rational numbers Q forms a group under the binary operation * defined by a * b = a + b - ab, for all a, b Є Q. Is this group abelian?

Answer:
(Q, *) will form a group if '*' satisfies these four properties:

(i) Closure property: Let a, b $\in$ Q then a*b = a + b – ab $\in$ Q as addition and multiplication of rational numbers will also be rational number.

(ii) Associative property: It says a*(b*c) = (a*b)*c, for all a, b, c $\in$ Q. Then according to the given definition of '*'

a*(b*c) = a*(b + c – bc) = a + (b + c – bc) – a(b + c – bc)
= a + b + c – ab – ac – bc – abc ……(1)

and (a*b)*c = (a + b – ab)*c = a + b – ab + c – (a + b – ab)c
= a + b + c – ab – ac – bc – abc …..(2)

By (1) and (2) associative property is satisfied.

(iii) Identity: Let e be the identity, then a*e = a
a + e – a.e = a (by definition of '*')
then e = 0 and 0 is s rational number.
Hence 0 is the identity.

(iv) Inverse: Let b be the inverse of a such that a*b = e = 0
Then a + b – ab = 0
b = - a / (1 – a), since a $\in$ Q, - a / (1 – a) $\in$ Q

Hence for all a $\in$ Q, there exist inverse of a in Q such that $a*a^{-1}$ =e.

Thus (Q, *) forms a group. Now for abelian group we have to check for commutative property.

Commutative property: It says, for a, b $\in$ Q a*b = b*a
Then a + b – ab = b + a – ba (addition and multiplication are always commutative)
Hence the given set Q with binary operation * defined as a*b = a + b – ab forms an abelian group.

b. How many generators are there of the cyclic group of order 8?

Answer:
Let a be the generator of G. Then *o*(a) = 8.

We can write G = {a, a^2, a^3, a^4, a^5, a^6, a^7, a^8}. We know that if G is cyclic group generated by a and o(a) = n, then a^m is a generator of G if and only if m and n are relatively prime. So we make the following observations:

7 is prime to 8, so a^7 is also a generator of G.
5 is prime to 8, so a^5 is also a generator of G.
3 is prime to 8, so a^3 is also a generator of G.

Since 2 and 8, 4 and 8, 6 and 8, 8 and 8 are not relatively prime; therefore none of the elements a^2, a^4, a^6 and a^8 can be generator of G.
Hence there are only four generators of G i.e; a, a^3, a^5 and a^7

Q.9 a. Determine the group code (3, 6) using parity check Matrix H given by

$$\mathbf{H} = \begin{bmatrix} 1\,0\,0 \\ 0\,1\,1 \\ 1\,1\,1 \\ 0\,1\,0 \\ 0\,0\,1 \end{bmatrix}$$

Answer:

Here the message code consisting 3 information digit is

B = [000, 100, 010, 001, 110, 011, 111]

The generator matrix G of the given parity check matrix is

$$G = \begin{bmatrix} 1\,0\,0\,1\,0\,0 \\ 0\,1\,0\,0\,1\,1 \\ 0\,0\,1\,1\,1\,1 \end{bmatrix}$$

So, we find the (3, 6) code C by $C = B^T.G$

$$\text{Thus, } C = \begin{bmatrix} 0\,0\,0 \\ 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \\ 1\,1\,0 \\ 1\,0\,1 \\ 0\,1\,1 \\ 1\,1\,1 \end{bmatrix} . \begin{bmatrix} 1\,0\,0\,1\,0\,0 \\ 0\,1\,0\,0\,1\,1 \\ 0\,0\,1\,1\,1\,1 \end{bmatrix} = \begin{bmatrix} 0\,0\,0\,0\,0\,0 \\ 1\,0\,0\,1\,0\,1 \\ 0\,1\,0\,0\,1\,1 \\ 0\,0\,1\,1\,1\,1 \\ 1\,1\,0\,1\,1\,0 \\ 1\,0\,1\,0\,1\,0 \\ 0\,1\,1\,1\,0\,0 \\ 1\,1\,1\,0\,0\,0 \end{bmatrix}$$

C is the desired code word.

b. Define Ring. Prove that if a, b $\in$(R, +, •), then $(a + b)^2 = a^2 + a•b + b•a + b^2$, where by x^2 we mean x•x.

Answer:

Ring: A non-empty set R with two operations called 'addition' and 'multiplication' is call a ring (R, +, .) if it satisfies the following axioms:

(i) (R, +) is an abelian group, that is

i. '+' is closed. If a, b $\in$R, then a + b$\in$R.
ii. '+' is associative i.e, for any a, b, c$\in$R, (a+b)+c = a+(b+c)
iii. '+' is commutative i.e, (a+b) = (b+a)
iv. There exist a additive identity e = 0 such that a+0 = 0+a = a
v. For a$\in$R, there exist -a$\in$R such that a+(-a) = (-a)+a = 0.

(ii) (R, .) is a semi-group i.e,

(a) '.' is closed i.e, for any a, b $\in$ R, a.b $\in$ R

vi. '.' is associative i.e, for any a, b, c $\in$ R, (a.b).c = a.(b.c)

(iii) Multiplication is distributive over addition i.e,

a.(b+c) = a.b + a.c and (a +b).c = a.c + b.c

Now it is given that $x^2 = x.x$

Thus $(a + b)^2 = (a + b).(a + b) = a.(a + b) + b.(a + b)$ by distributive law

$= (a.a + a.b) + (b.a + b.b)$

$= (a^2 + a.b + b.a + b^2)$ given that $x^2 = x.x$

Hence proved.

TEXT BOOK

Discrete Mathematical Structures, D.S. Chandrasekharaiah, Prism Books Pvt. Ltd., 2005.